Remarks on double zeta values of level 2

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Abstract

We give a generator of the space spanned by double zeta values of level 2 with odd weight by using explicit formulas for double Euler sums.

1 Introduction and main results

Double zeta values of level 2 are defined by

$$\zeta^{\mathbf{o}}(r,s)=\sum_{\substack{m>n>0\\ m,n:\text{odd}}}\frac{1}{m^r n^s}, \tag{1}$$

where r,s are positive integers with $r\geq 2$. These real values can be regarded as a kind of Euler sums (see Section 2) or multiple Hurwitz series (see [8]), which are well-studied, but apparently it is believed that the form (1) matches the theory of modular forms. The relationship between double zeta values and modular forms was originally studied in [5]. As a consequence of their study, Kaneko and Tasaka [7] considered the case of level 2 involved "double Eisenstein series", and they found an explicit connection between modular forms of $\Gamma_0(2)$ and (1), when weight ($=r+s$) is even.

In the present paper, we mainly discuss the case of odd weight. Let $\mathcal{DO}_k$ be the $\mathbb{Q}$-vector space spanned by double zeta values of level 2 and weight k, namely,

$$\mathcal{DO}_k=\langle\zeta^{\mathbf{o}}(r,k-r)\mid 2\leq r\leq k-1\rangle_{\mathbb{Q}}.$$

We first introduce our result about a generator of the space $\mathcal{DO}_k$.

Theorem 1. *For odd $k\geq 3$, the $(k+1)/2$ numbers $\{(\log 2)\pi^{k-1},\zeta(k-2i)\pi^{2i}\mid 0\leq i\leq (k-3)/2\}$ span the same space as the space $\mathcal{DO}_k$.*

We remark that Theorem 1 can be viewed as the level 2 version of Zagier's result ([11, Theorem 2]). He proved that, for odd $k\geq 5$, the $\mathbb{Q}$-vector space $\mathcal{DZ}_k$ generated by double zeta values $\zeta(r,s)=\sum_{m>n>0}m^{-r}n^{-s}$ of weight k has the generator $\{\zeta(k-2i)\pi^{2i}\mid 0\leq i\leq (k-3)/2\}$, which we believe being a base. He also showed that the $(k-1)/2$ numbers $\zeta(k-1-2i,2i+1)$ $(0\leq i\leq (k-3)/2)$ satisfy $\dim S_{k-1}(1)+\dim S_{k+1}(1)=[(k-11)/6]$ $\mathbb{Q}$-linear relations (see [11, Theorem 3]), where $S_k(1)$ is the space of cusp forms of weight k on $\mathrm{SL}_2(\mathbb{Z})$. The same discussion for our case is given in the following theorem.

Theorem 2. *For odd $k \geq 5$, the $\mathbb{Q}$-vector space spanned by $\zeta^{\mathbf{o}}(i, k-i)$ $(2 \leq i \leq k-2)$ is the same space as $\mathcal{DZ}_k$.*

Theorem 2 implies that the $k-3$ numbers $\zeta^{\mathbf{o}}(i, k-i)$ $(2 \leq i \leq k-2)$ satisfy $\dim S_{k-1}(2) + \dim S_{k+1}(2) = (k-5)/2$ $\mathbb{Q}$-linear relations, where $S_k(2)$ is the space of cusp forms of weight k on $\Gamma_0(2)$. However, there are no direct connection between "period polynomials" and double zeta values of level 2 in the meaning as in $\mathrm{SL}_2(\mathbb{Z})$ (see [11, Section 6]).

In Section 2, we give proofs of Theorems 1 and 2 by using explicit formulas of double zeta values of level 2 which follows from evaluations of double Euler sums. The last section we present some remarks on the sum formula of double zeta values of level 2.

2 Euler sums and proof of Theorem 1

Now we define the double Euler sums by

$$\zeta(r,s) = \sum_{m>n>0} \frac{1}{m^r n^s} \ (r \geq 2, s \geq 1), \quad \zeta(r,\overline{s}) = \sum_{m>n>0} \frac{(-1)^n}{m^r n^s} \ (r \geq 2, s \geq 1),$$
$$\zeta(\overline{r},s) = \sum_{m>n>0} \frac{(-1)^m}{m^r n^s} \ (r,s \geq 1), \quad \zeta(\overline{r},\overline{s}) = \sum_{m>n>0} \frac{(-1)^{n+m}}{m^r n^s} \ (r,s \geq 1).$$

(Each ranges of r, s give convergence conditions of each double series.) These real values have a deep connection with knot theory and quantum field theory (e.g. [2, 3]), and many studies have been done. In [3], they conjectured about the number of (algebra) generators of the space of Euler sums. Let $\mathcal{DL}_k$ be a $\mathbb{Q}$-vector space spanned by double Euler sums of weight k. We can easily deduce that Broadhurst-Kreimer conjecture involved double Euler sums says

$$\dim_{\mathbb{Q}} \mathcal{DL}_k = \left[\frac{k+1}{2}\right] \quad (k \geq 2).$$

For odd $k > 2$, we can prove that the space $\mathcal{DL}_k$ is spanned by the set $\{(\log 2)\pi^{k-1}, \zeta(k-2i)\pi^{2i} \mid 0 \leq i \leq (k-3)/2\}$ by using the following closed formulas for double Euler sums (see for example [10, (4) and (5)]). Let $\zeta(\overline{k}) = \sum_{n>0}(-1)^n n^{-k}$. We note that $\zeta(\overline{1}) = -\log 2$ and $\zeta(\overline{k}) = (2^{1-k}-1)\zeta(k)$ for $k \geq 2$.

Proposition 3. *For odd $k \geq 3$ and positive integers r, s with $r + s = k$, double Euler sums* *are given in terms of products* $\zeta(2i)\zeta(k-2i), \zeta(\overline{2i})\zeta(k-2i)$ *and* $\zeta(\overline{2i})\zeta(\overline{k-2i})$ *as follows:*

$$2\zeta(\overline{r},\overline{s}) = \zeta(\overline{r})\zeta(\overline{s}) - \zeta(k) - (-1)^s\zeta(\overline{r})\zeta(\overline{s}) + (-1)^s\Bigg\{ - \binom{k-1}{r}\zeta(\overline{k})$$
$$- \binom{k-1}{s}\zeta(\overline{k}) + 2\sum_{j=1}^{[r/2]}\binom{k-2j-1}{r-2j}\zeta(\overline{k-2j})\zeta(2j)$$
$$+ 2\sum_{j=1}^{[s/2]}\binom{k-2j-1}{s-2j}\zeta(\overline{k-2j})\zeta(2j)\Bigg\} \quad (\textit{for all } r,s \geq 1), \tag{2}$$

$$2\zeta(r,\overline{s}) = \zeta(r)\zeta(\overline{s}) - \zeta(\overline{k}) - (-1)^s\zeta(r)\zeta(\overline{s}) + (-1)^s\Bigg\{ - \binom{k-1}{r}\zeta(\overline{k})$$
$$- \binom{k-1}{s}\zeta(k) + 2\sum_{j=1}^{[r/2]}\binom{k-2j-1}{r-2j}\zeta(\overline{k-2j})\zeta(\overline{2j})$$
$$+ 2\sum_{j=1}^{[s/2]}\binom{k-2j-1}{s-2j}\zeta(k-2j)\zeta(\overline{2j})\Bigg\} \quad (\textit{for all } r \geq 2, s \geq 1), \tag{3}$$

$$2\zeta(\overline{r},s) = \zeta(\overline{r})\zeta(s) - \zeta(\overline{k}) - (-1)^s\zeta(\overline{r})\zeta(s) + (-1)^s\Bigg\{ - \binom{k-1}{r}\zeta(k)$$
$$- \binom{k-1}{s}\zeta(\overline{k}) + 2\sum_{j=1}^{[r/2]}\binom{k-2j-1}{r-2j}\zeta(k-2j)\zeta(\overline{2j})$$
$$+ 2\sum_{j=1}^{[s/2]}\binom{k-2j-1}{s-2j}\zeta(\overline{k-2j})\zeta(\overline{2j})\Bigg\} \quad (\textit{for all } r \geq 1, s \geq 2). \tag{4}$$

The exceptional case of (4)*, which is* $s = 1$*, can be written as follows:*

$$2\zeta(\overline{r},1) = \zeta(r+1) + (r-1)\zeta(\overline{r+1}) - 2\sum_{j=1}^{r/2-1}\zeta(r+1-2j)\zeta(\overline{2j}). \tag{5}$$

Zagier showed that the double zeta value $\zeta(r,s)$ with $r+s=k$ (k:odd) can be expressed as $\mathbb{Q}$-linear combinations of two products $\zeta(2i)\zeta(k-2i)$ $(0 \leq i \leq (k-3)/2)$ (see [11, Proposition 7]), using his method which is based on the double shuffle relations and the theory of generating functions. By his results and Proposition 3, the space spanned by $\{(\log 2)\pi^{k-1}, \zeta(k-2i)\pi^{2i} \mid 0 \leq i \leq (k-3)/2\}$ contains $\mathcal{DL}_k$. Conversely, using shuffle products (see [1]), we can easily check that two products of $\zeta(2i)\zeta(\overline{k-2i})$ $(0 \leq i \leq (k-1)/2)$ are in the space $\mathcal{DL}_k$. Then we have the following theorem.

Theorem 4. *For odd* $k \geq 3$*, the* $(k+1)/2$ *numbers* $\{(\log 2)\pi^{k-1}, \zeta(k-2i)\pi^{2i} \mid 0 \leq i \leq (k-3)/2\}$ *span the same space as* $\mathcal{DL}_k$*.*

Now we begin considering the case of double zeta values of level 2 and shall give a proof of Theorem 1. For $r \geq 2$ and $s \geq 1$, it is easily seen that

$$\zeta^{\mathbf{o}}(r,s) = \frac{1}{4}(\zeta(r,s) - \zeta(\overline{r},s) - \zeta(r,\overline{s}) + \zeta(\overline{r},\overline{s})), \tag{6}$$

and this shows that $\mathcal{DL}_k$ contains $\mathcal{DO}_k$. (However, more precisely, one can find $\mathcal{DL}_k = \mathcal{DO}_k$ from Theorem 1 and Theorem 4.) Using (2), (3), (4), (5) and (6), we can easily obtain the explicit formulas for double zeta values of level 2.

Proposition 5. *For odd $k \geq 5$ and $k = r + s$ $(r, s \geq 2)$, one has*

$$\begin{aligned} 2\zeta^{\mathbf{o}}(r,s) = &-\zeta^{\mathbf{o}}(k) + (1 - (-1)^s)\zeta^{\mathbf{o}}(r)\zeta^{\mathbf{o}}(s) \\ &+ 2(-1)^s \sum_{j=1}^{\max([r/2],[s/2])} \left(\binom{k-2j-1}{r-2j} + \binom{k-2j-1}{s-2j} \right) \zeta^{\mathbf{e}}(k-2j)\zeta^{\mathbf{o}}(2j), \end{aligned} \tag{7}$$

where $\zeta^{\mathbf{e}}(k) = \sum_{n>0}(2n)^{-k}$. Furthermore, for $a \geq 1$, we have

$$2\zeta^{\mathbf{o}}(2a,1) = -2\sum_{j=1}^{a-1} \zeta^{\mathbf{e}}(2a-2j+1)\zeta^{\mathbf{o}}(2j) - 2\zeta(\overline{1})\zeta^{\mathbf{o}}(2a) - \zeta^{\mathbf{o}}(2a+1)). \tag{8}$$

Our strategy to prove Theorem 1 is to find a basis of the space $\mathcal{DO}_k$. To do this, now we give following two lemmas.

Lemma 6. *For odd $k \geq 3$, the $(k-1)/2$ numbers $\{\zeta(k-2i)\pi^{2i} \mid 0 \leq i \leq (k-3)/2\}$ span the same space as the $(k-1)/2$ numbers $\{\zeta^{\mathbf{o}}(k-2r,2r), \zeta^{\mathbf{o}}(k) \mid 1 \leq r \leq (k-3)/2\}$.*

Proof. Using (7), for odd $k \geq 5$, one has

$$\begin{aligned} &\zeta^{\mathbf{o}}(k-2r,2r) \\ &= -\frac{1}{2}\zeta^{\mathbf{o}}(k) + \sum_{j=1}^{(k-3)/2} \left(\binom{k-2j-1}{k-2r-1} + \binom{k-2j-1}{2r-1} \right) \zeta^{\mathbf{e}}(k-2j)\zeta^{\mathbf{o}}(2j). \end{aligned}$$

Let M_k be the $(k-1)/2 \times (k-1)/2$ matrix whose coefficients are

$$m_{rj} = \begin{cases} \binom{k-2j-1}{k-2r-1} + \binom{k-2j-1}{2r-1} & 1 \leq r,j \leq (k-3)/2, \\ \delta_{r,j} & \text{others}, \end{cases}$$

where $\delta_{r,j}$ is Kronecker's delta. Surprisingly, the matrix M_k is exactly equal to $\mathcal{A}$ of [11, Lemma 3] excepting 1-th row and 1-th column of M_k, and then it has non-zero determinant. This induces the following equality:

$$\begin{aligned} &\langle \zeta^{\mathbf{o}}(k), \zeta^{\mathbf{o}}(k-2i,2i) \mid 1 \leq i \leq (k-3)/2 \rangle_{\mathbb{Q}} \\ &= \langle \zeta^{\mathbf{o}}(k), \zeta^{\mathbf{e}}(k-2i)\zeta^{\mathbf{o}}(2i) \mid 1 \leq i \leq (k-3)/2 \rangle_{\mathbb{Q}}, \end{aligned} \tag{9}$$

which completes the proof of Lemma 6. □

Note that since $\zeta^{\mathbf{o}}(k)=(1-2^{-k})\zeta(k)$ and $\zeta^{\mathbf{e}}(k)=2^{-k}\zeta(k)$ for $k\geq 2$, the right hand-side of (9) is equal to the space $\mathcal{DZ}_k$.

Lemma 7. *For each odd $k\geq 3$, $\zeta^{\mathbf{o}}(k)$ can be expressed as $\mathbb{Q}$-linear combinations of $\zeta^{\mathbf{o}}(r,k-r)$ $(2\leq r\leq k-2)$.*

Proof. It is easy to check the following

$$\begin{aligned}&\zeta^{\mathbf{o}}(2,k-2)\\&=-\frac{1}{2}\zeta^{\mathbf{o}}(k)+\zeta^{\mathbf{o}}(2)\zeta^{\mathbf{o}}(k-2)-\sum_{j=1}^{(k-3)/2}\frac{k-2j-1+\delta_{j,1}}{2^{k-2j}-1}\zeta^{\mathbf{o}}(k-2j)\zeta^{\mathbf{o}}(2j).\end{aligned}$$

Using $\zeta^{\mathbf{o}}(a)\zeta^{\mathbf{o}}(b)=\zeta^{\mathbf{o}}(a,b)+\zeta^{\mathbf{o}}(b,a)+\zeta^{\mathbf{o}}(a+b)$, we have

$$\begin{aligned}&\zeta^{\mathbf{o}}(k)\Big(-\frac{1}{2}+\sum_{j=1}^{(k-3)/2}\frac{k-2j-1+\delta_{j,1}}{2^{k-2j}-1}\Big)\\&=\zeta^{\mathbf{o}}(k-2,2)-\sum_{j=1}^{(k-3)/2}\frac{k-2j-1+\delta_{j,1}}{2^{k-2j}-1}(\zeta^{\mathbf{o}}(k-2j,2j)+\zeta^{\mathbf{o}}(2j,k-2j)).\end{aligned}$$

The coefficient of $\zeta^{\mathbf{o}}(k)/2$

$$2\Big(-\frac{1}{2}+\sum_{j=1}^{(k-3)/2}\frac{k-2j-1+\delta_{j,1}}{2^{k-2j}-1}\Big)$$

are non-zero, because it is a 2-adic unit, which completes the proof. □

Proof of Theorem 1. Let $\mathcal{DO}_k^{\mathbf{ev}}$ be the space spanned by $\{\zeta^{\mathbf{o}}(k-2r,2r),\zeta^{\mathbf{o}}(k-1,1),\zeta^{\mathbf{o}}(k)\mid 1\leq r\leq (k-3)/2\}$, so that,

$$\mathcal{DO}_k^{\mathbf{ev}}=\langle\zeta^{\mathbf{o}}(k-2r,2r),\zeta^{\mathbf{o}}(k-1,1),\zeta^{\mathbf{o}}(k)\mid 1\leq r\leq (k-3)/2\rangle_{\mathbb{Q}}.$$

From Lemma 6 and (8), we have

$$\langle(\log 2)\pi^{k-1},\zeta(3)\pi^{k-3},\dots,\zeta(k-2)\pi^2,\zeta(k)\rangle_{\mathbb{Q}}=\mathcal{DO}_k^{\mathbf{ev}}.$$

On the other hand, by Proposition 5 and Lemma 7, one can find

$$\langle(\log 2)\pi^{k-1},\zeta(3)\pi^{k-3},\dots,\zeta(k-2)\pi^2,\zeta(k)\rangle_{\mathbb{Q}}\supset\mathcal{DO}_k\supset\mathcal{DO}_k^{\mathbf{ev}},$$

which completes the proof of Theorem 1. □

Proof of Theorem 2. Let $\mathcal{DO}_k^*$ be the space spanned by $\zeta^{\mathbf{o}}(r,k-r)$ $(2\leq r\leq k-2)$. By Proposition 5 and Lemma 6, the space generated by $\{\zeta(k-2i)\pi^{2i}\mid 0\leq i\leq (k-3)/2\}$ is the same space as the space $\mathcal{DO}_k^*$, and this induces $\mathcal{DO}_k^*=\mathcal{DZ}_k$, which completes the proof of Theorem 2. □

3 Remarks on the sum formula

When k is even, Kaneko and Tasaka found the relationship between double zeta values of level 2 and modular forms (see [7, Corollary p.17]), and they also showed the following sum formula (see [7, Theorem 1]), using their 'double shuffle relation'.

Proposition 8. *For even $k \geq 4$, we have*

$$\sum_{r=1}^{k/2-1} \zeta^{\mathbf{o}}(2r, k-2r) = \frac{1}{4}\zeta^{\mathbf{o}}(k). \tag{10}$$

Proposition 8 can be extended to certain symmetric sums of multiple zeta values of level 2 by using Hoffman's 'harmonic product' (see [6, Theorem 2.2]). In this section, we give another proof of Proposition 8 using certain properties of the Bernoulli polynomials.

We denote by $B_n(x)$ the n-th Bernoulli polynomial defined by

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \qquad |t| < 2\pi,$$

and the n-th Bernoulli number B_n by $B_n := B_n(0)$. Euler proved the next equations;

Lemma 9. *For $n \geq 1$, we have*

$$\zeta^{\mathbf{o}}(2n) = (1 - 2^{-2n})\zeta(2n) \quad \text{and} \quad \zeta(2n) = -\frac{(-4\pi^2)^n}{2(2n)!}B_{2n}. \tag{11}$$

Lemma 10 (see [4, p. 4, 6 and 119])**.** *We have the following properties;*

$$B_{2n-1} = 0, \qquad n \geq 2, \tag{12}$$

$$B_n(1/2) = -(1 - 2^{1-n})B_n, \tag{13}$$

$$\sum_{m=0}^{n} \binom{n}{m} B_m(x)B_{n-m}(y) = n(x+y-1)B_{n-1}(x+y) - (n-1)B_n(x+y). \tag{14}$$

Proof of Proposition 8. This is an analogue of the proof of [9, Theorem A]. Put $x = y = 1/2$ in (14) and use (12) and (13). Then we have

$$\begin{aligned}
&\sum_{m=0}^{n} \binom{n}{m} B_m(1/2)B_{n-m}(1/2) \\
&= \sum_{m=0}^{n} \binom{n}{m} (1 - 2^{1-m})(1 - 2^{1-n+m})B_m B_{n-m} = -(n-1)B_n.
\end{aligned}$$

Obviously, one has

$$\begin{aligned}
&(1 - 2^{1-m})(1 - 2^{1-n+m}) = 2(1 - 2^{-m})(1 - 2^{-n+m}) + 2(2^{-m} - 1) + 1, \\
&\sum_{m=0}^{2n} \binom{2n}{m} B_m B_{2n-m} = -(2n-1)B_{2n}.
\end{aligned}$$

Therefore it holds that

$$\sum_{m=2}^{2n-2} 2(1-2^{-m})(1-2^{-n+m})\binom{2n}{m} B_m B_{2n-m} = (2^{-2n}-1)(2n-1)B_{2n}.$$

By using (11), we obtain

$$\sum_{m=1}^{n-1} \zeta^{\mathbf{o}}(2m)\zeta^{\mathbf{o}}(2n-2m) = \frac{2n-1}{2}\zeta^{\mathbf{o}}(2n). \tag{15}$$

On the other hand, by the harmonic product formula, it holds that

$$\zeta^{\mathbf{o}}(2m)\zeta^{\mathbf{o}}(2n-2m) = \zeta^{\mathbf{o}}(2m, 2n-2m) + \zeta^{\mathbf{o}}(2n-2m, 2m) + \zeta^{\mathbf{o}}(2n).$$

By summing the above formula on m from 1 to $n-1$, we have

$$\sum_{m=1}^{n-1} \zeta^{\mathbf{o}}(2m)\zeta^{\mathbf{o}}(2n-2m) = 2\sum_{m=1}^{n-1} \zeta^{\mathbf{o}}(2m, 2n-2m) + (n-1)\zeta^{\mathbf{o}}(2n). \tag{16}$$

Hence we obtain Proposition 8 by (15) and (16). □

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